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On a Certain Subset of $L_1(0, 1)$ and Non-Existence of Best Approximation in Some Spaces of Operators

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We prove that there is a bounded linear operator $T: l_\infty \to l_\infty$ for which there is no closest compact linear $K: l_\infty \to l_\infty$. A similar result is proved for an operator from l_1 to $L_1(0, 1)$. This implies, in the cases where $X = L_1(0, 1)$, $X = L_\infty(0, 1)$ or $X = C(0, 1)$, that there is an operator $T: X \to X$ with no best compact approximation (see Appendix). We show that $L_1(0, 1)$ contains a set S, bounded and non-empty, for which $\inf_C \sup_{x \in S} d(x, C)$ over all compact subsets C of $L_1(0, 1)$ is not attained. The set S is used to prove the above results.

1. Introduction

Several authors have considered the problem of determining the pairs of Banach spaces E, F for which $C(E, F)$ is proximinal in $L(E, F)$. (See Section 2 for definitions and notations.) In particular, $C(E, E)$ is known to be proximinal in $L(E, E)$ when $E = l_p$, $1 \leqslant p < \infty$, or $E = c_0$. (See [1, 3–7, 9, 11]. [1] contains a detailed historical survey.) However, we prove:

THEOREM 1. *$C(l_\infty, l_\infty)$ is not a proximinal subspace of $L(l_\infty, l_\infty)$.*

It seems that the above is the first example of a classical Banach space E for which $C(E, E)$ is not proximinal in $L(E, E)$.

For a bounded set S in a Banach space E we consider the infimum

$$a(S) = \inf_C \sup_{x \in S} d(x, C), \tag{1.1}$$

where C ranges over all compact subsets of E. $a(S)$ is called in [9] "the Kuratowski measure of non-compactness" of S. It is easy to see that $a(S)$ is the infimum of the set of all $r > 0$ for which there is a finite r-net of S. We prove:

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THEOREM 2. *The space $L_1(0, 1)$ has a bounded subset S such that for every compact subset C of $L_1(0, 1)$ there is $x \in S$ with $d(x, C) > a(S)$. (That is, the infimum* (1.1) *is not attained.)*

Theorem 2 will be proved in Section 3 and will be used in Section 4 to prove Theorem 1. We will also prove the following.

THEOREM 3. *$C(l_1, L_1(\mu))$ is proximinal in $L(l_1, L_1(\mu))$ if and only if μ is purely atomic.*

2. DEFINITIONS, NOTATIONS, AND CONVENTIONS

An "operator" in this paper is always bounded and linear. When E and F are Banach spaces, $L(E, F)$ denotes the Banach space of all operators from E to F, with the sup norm. $C(E, F)$ is the subspace of $L(E, F)$ of the compact operators. When E and F are fixed, the essential norm (introduced in [1]) of $T \in L(E, F)$, $\| T \|_e$, is defined by

$$\| T \|_e = \inf\{\| T - K \| : K \in C(E, F)\} = d(T, C(E, F)).$$

$\| T \|_e$ depends on F—if F is a subspace of G and $J: F \to G$ is the inclusion, $\| JT \|_e$ may be smaller than $\| T \|_e$ (cf. [1a]). By a "set" here we mean a non-empty set. A set S in a metric space (X, d) is said to be proximinal if for every x in X there is y in S so that $d(x, y) = d(x, S)$, where $d(x, S) = \inf\{d(x, z): z \in S\}$. When A and B are sets in a vector space and λ is a scalar, $A + B$ is the set of all sums $x + y$, $x \in A$, $y \in B$ and λA is the set of all λ multiples of elements of A. Finally, for a Banach space E, B_E denotes the closed unit ball of E.

3. A SUBSET OF $L_1(0, 1)$

In this section we prove Theorem 2. Let $I: L_1(0, 1) \to L_1(0, 1)$ be the identity operator. For positive integers n and i we define the closed interval $\Delta_i^{(n)} = [(i - 1) 2^{-n}, i2^{-n}]$. For a set A, the characteristic function χ_A is defined by $\chi_A(x) = 1$ when $x \in A$ and $\chi_A(x) = 0$ otherwise. The operators

$$P_n : L_1(0, 1) \to L_1(0, 1)$$

defined by

$$P_n(f) = \sum_{i=1}^{2^n} \left(2^n \int_{\Delta_i^{(n)}} f(t)\, dt\right) \chi_{\Delta_i^{(n)}}$$

are contractive projections and $P_n \to I$ uniformly on every compact set.

We now define a set $S \subset L_1(0, 1)$ as follows: put $B = B_{L_1(0,1)}$, $C_n = P_n(B)$ and

$$S = \bigcap_{n=1}^{\infty} \left[3C_n + \left(1 + \frac{1}{n}\right) B\right]. \tag{3.1}$$

The Lebesgue measure will be denoted by λ.

LEMMA 1. *Let $[a, b] = I_1 \cup I_2 \cup \cdots \cup I_{2n}$ be a decomposition of the closed interval $[a, b]$ into $2n$ subintervals so that $\lambda I_i = (2n)^{-1}(b - a)$ for every i (i.e., all I_i have same length). Let A be a measurable set such that*

$$\begin{aligned} \lambda(A \cap I_i) &= \theta(2n)^{-1}(b - a), \qquad & i = 1, 3, \ldots, 2n - 1, \\ &= (1 - \theta)(2n)^{-1}(b - a), \qquad & i = 2, 4, \ldots, 2n, \end{aligned}$$

for some $\frac{1}{2} < \theta < 1$. Then for every two constants α and k and every step function $g = \sum_{i=1}^{2n} \alpha_i \chi_{I_i}$

$$\int_a^b |\alpha\chi_A - k - g|\, dt \geqslant \int_a^b |\alpha\chi_A|\, dt - (2\theta - 1) \int_a^b |g|\, dt.$$

Proof. It follows that $\lambda(A \cap [a, b]) = (b - a)/2$. Now,

$$\begin{aligned} &\int_a^b |\alpha\chi_A - k - g|\, dt \\ &\quad = \frac{b-a}{2n} \sum_{j=1}^{n} \{\theta\, |\alpha - k - \alpha_{2j-1}| + (1 - \theta)\, |k + \alpha_{2j-1}| \\ &\qquad + (1 - \theta)\, |\alpha - k - \alpha_{2j}| + \theta\, |k + \alpha_{2j}|\} \\ &\quad \geqslant \frac{b-a}{2n} \sum_{j=1}^{n} \{|\alpha| - (2\theta - 1)(|\alpha_{2j-1}| + |\alpha_{2j}|)\} \\ &\quad = \frac{b-a}{2} |\alpha| - (2\theta - 1) \frac{b-a}{2n} \sum_{i=1}^{2n} |\alpha_i| \\ &\quad = \int_a^b |\alpha\chi_A|\, dt - (2\theta - 1) \int_a^b |g|\, dt. \end{aligned}$$

Proof of Theorem 2. We will show that the set S defined in (3.1) satisfies:

(a) $a(S) = 1$.

(b) For every compact set K in $L_1(0, 1)$ there is an x in S such that $d(x, K) > 1$.

Since $S \subset (3C_n) + (1 + 1/n) B$ for every n, $a(S) \leqslant 1 + 1/n$ for every n. Hence $a(S) \leqslant 1$. Since (b) implies $a(S) \geqslant 1$, it is sufficient to prove (b).

Let K be a compact subset of $L_1(0, 1)$. For some $\beta > 0$, $K \subset \beta B$. The projections P_n converge uniformly to I on compact sets. Thus, for a suitable integer $p > 1$, $\|f - P_p f\| \leqslant \frac{1}{8}$ for every $f \in K$. Let K_1 be the compact set $K_1 = (I - P_p) K$. Now define $\alpha = \alpha(p) > 2$ and $\epsilon = \epsilon(p) > 0$ by

$$\alpha = 2\left(1 + \frac{1}{p}\right) \tag{3.2}$$

$$\epsilon = \frac{1}{4}\left(\alpha + \frac{1}{2\alpha} - \frac{9}{4}\right) \tag{3.3}$$

($\alpha > 2$ yields $\epsilon > 0$ easily). There exists an integer q, $q > p$, so that $(I - P_q) K_1 \subset \epsilon B$. Thus,

$$K \subset \beta C_p + \tfrac{1}{8} C_q + \epsilon B. \tag{3.4}$$

Let θ be defined by

$$\theta = 1 - 1/\alpha. \tag{3.5}$$

Note that $\frac{1}{2} < \theta < 1$.

Now construct a set A as follows. First define

$$\begin{aligned} a_i &= \theta &&\text{if} \quad 0 \leqslant i (\operatorname{mod} 2^{q-p}) < 2^{q-p-1}, \\ &= 1 - \theta &&\text{if} \quad 2^{q-p-1} \leqslant i (\operatorname{mod} 2^{q-p}) < 2^{q-p} \end{aligned}$$

($i(\operatorname{mod} n) = j$ if $0 \leqslant j < n$ and n divides $i - j$). Then

$$A = \bigcup_{i=1}^{2^q} [(i-1)\, 2^{-q}, (i-1)\, 2^{-q} + a_i 2^{-q}]. \tag{3.6}$$

A has the following two properties:

(i) $\lambda(A \cap \Delta_j^{(p)}) = \frac{1}{2}\lambda(\Delta_j^{(p)}) = 2^{-p-1}$ ($j = 1, 2, 3, \ldots, 2^p$).

(ii) For every $j = 1, 2, 3, \ldots, 2^p$ and every $n = p + 1, p + 2, p + 3, \ldots, q$, $\lambda(A \cap \Delta_i^{(n)}) = \theta\lambda(\Delta_i^{(n)}) = \theta 2^{-n}$ for exactly 2^{n-p-1} of the 2^{n-p} indices i satisfying $\Delta_i^{(n)} \subset \Delta_j^{(p)}$ and $\lambda(A \cap \Delta_i^{(n)}) = (1 - \theta)\, 2^{-n}$ for the remaining 2^{-n-p-1}.

Now put $f = \alpha\chi_A$. Since $\lambda(A) = \frac{1}{2}$, $\|f\| = |\alpha|\, \lambda(A) = 1 + 1/p$. Hence $f \in 3C_n + (1 + 1/n) B$ for $n = 1, 2, 3, \ldots, p$. $g \in 3C_{p+1}$ defined by $g = \alpha \sum_{i=1}^{2^p} \chi_{\Delta_{2i-1}^{(p+1)}}$ satisfies $\|f - g\| = \alpha(1 - \theta) = 1$. Hence $f \in 3C_{p+1} + B$ and clearly $f \in 3C_n + (1 + 1/n) B$ for all $n > p$ (since $C_n \supset C_{p+1}$). From the above, $f \in 3C_n + (1 + 1/n) B$ for all n, i.e., $f \in S$.

It is left to show that $d(f, K) > 1$. By (3.4), it is sufficient to prove $d(f, \beta C_p + \frac{1}{8} C_q) > 1 + \epsilon$. Let $h \in \beta C_p$ and $g \in \frac{1}{8} C_q$. For every

$i = 1, 2, 3,..., 2^p$ h is constant on $\Delta_i^{(p)}$. Lemma 1 may now be used with $[a, b] = \Delta_i^{(p)}$, $n = 2^{q-p-1}$, $\{I_1, I_2,..., I_{2n}\}$ a suitable renumbering of the $\Delta_j^{(q)}$'s contained in $\Delta_i^{(p)}$ and k the value of h on $\Delta_i^{(p)}$. We get

$$\int_{\Delta_i^{(p)}} | \alpha\chi_A - h - g | \, dt \geqslant \int_{\Delta_i^{(p)}} | \alpha\chi_A | \, dt - (2\theta - 1) \int_{\Delta_i^{(p)}} | g | \, dt.$$

Summing over $i = 1, 2, 3,..., 2^p$ gives

$$\| f - h - g \| \geqslant \| f \| - (2\theta - 1) \| g \| \geqslant \| f \| - \frac{1}{8}(2\theta - 1)$$
$$= \frac{\alpha}{2} - \frac{1}{8}\left(2\left(1 - \frac{1}{\alpha}\right) - 1\right) = 1 + 2\epsilon.$$

4. Proof of Theorems 1 and 3

Denote by $\{e_i\}$ the natural unit vector basis of l_1.

Proposition 1. *Let E be a Banach space and let $T \in L(l_1, E)$ then*

1. $\| T \| = \operatorname{Sup}_i \| Te_i \|$,
2. $\| T \|_e = a(\{Te_i : i = 1, 2, 3,...\})$.

Proof. 1. It is an easy well-known fact. 2. Let $K \in C(l_1, E)$ then $d(Te_i, \overline{\{Ke_j\}_{j=1}^\infty}) \leqslant \| T - K \|$ for all i. Thus $a(\{Te_i\}_{i=1}^\infty) \leqslant \| T \|_e$. Let C be a compact set in E. For every j pick $y_j \in C$ such that $\| Te_j - y_j \| \leqslant \sup_i d(Te_i, C)$. The compact operator $K \in C(l_1, E)$ defined by $Ke_j = y_j$ satisfies $\| T - K \| \leqslant \sup_i d(Te_i, C)$. We get that $\| T \|_e \leqslant \sup_i d(Te_i, C)$ for every compact set C in E. Hence $\| T \|_e \leqslant a(\{Te_i\}_{i=1}^\infty)$.

Proposition 2. *The following are equivalent for a Banach space E.*

(a) *$C(l_1, E)$ is proximinal in $L(l_1, E)$.*

(b) *For every bounded countable set S in E the infimum* (1.1) *is attained.*

Proof. (a) $\Rightarrow$ (b): Suppose (a) holds and let $S = \{x_i : i = 1, 2, 3,...\}$ be a bounded countable set in E. Define $T \in L(l_1, E)$ by $Te_i = x_i$. Let $K \in C(l_1, E)$ be a closest compact operator to T (i.e., $\| T \|_e = \| T - K \|$) and put $C = \{Ke_i : i = 1, 2,...\}$. By Proposition 1, $a(S) = \| T \|_e = \| T - K \| \geqslant \sup d(x_i, \bar{C})$.

(b) $\Rightarrow$ (a): Let T be an element of $L(l_1, E)$. Assume (b) holds then there is a compact set C such that $\| T \|_e = a(\{Te_i\}_{i=1}^\infty) = \sup_i d(Te_i, C)$. Now if we define K as in Proposition 1, $\| T - K \| \leqslant \sup_i d(Te_i, C) = \| T \|_e$.

The following fact is well known and easy to prove.

PROPOSITION 3. *Let E and F be Banach spaces. The map α: $L(E, F^*) \to L(F, E^*)$ defined by $\alpha(T) = T^*J$, where J: $F \to F^{**}$ is the natural embedding, is a isometry onto. Furthermore, α maps $C(E, F^*)$ onto $C(F, E^*)$.*

Proof. Let $J_1 : E \to E^{**}$ be the natural embedding. Define β: $L(F, E^*) \to L(E, F^*)$ by $\beta(S) = S^*J_1$. It is easy to see now that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identities of $L(F, E^*)$ and $L(E, F^*)$, respectively. Since α and β are both contractive, the isometry claim is proved. α is surjective since it has a right side inverse. Finally, if T is compact, $T^*J = \alpha(T)$ is compact and if T^*J is compact, $T = (T^*J)^* J_1$ is also compact.

PROPOSITION 4. *$L_1(0, 1)$ is isometric to the range of a contractive projection in $(l_\infty)^*$.*

Proof. $L_\infty(0, 1)$ is isometric to the range of a contractive projection in l_∞ (see [10]). Hence, there is a contractive projection in $(l_\infty)^*$ whose range is isometric to $(L_\infty(0, 1))^*$. But it is known that there is a contractive projection in $(L_\infty(0, 1))^*$ with range isometric to $L_1(0, 1)$ (see [2, p. 163]).

PROPOSITION 5. *Let E and F be Banach spaces and let G be the range of a contractive projection in F. Then if $C(E, F)$ is promininal in $L(E, F)$, $C(E, G)$ is proximinal in $L(E, G)$.*

Proof. Trivial.

Proof of Theorem 1. By Theorem 2 there is a set S in $L_1(0, 1)$ for which the infimum (1.1) is not attained. Let S_0 be a countable dense subset of S. Clearly $a(S_0) \leqslant a(S) = 1$. For every compact set C in $L_1(0, 1)$ there is $x \in S$ with $d(x, C) > 1$. Since S_0 is dense in S, there exists $y \in S_0$ with $d(y, C) > 1$. Hence $a(S_0) = 1$ and the infimum (1.1) is not attained for S_0 as well. By Proposition 2, $C(l_1, L_1(0, 1))$ is not proximinal in $L(l_1, L_1(0, 1))$. By Propositions 4 and 5, this fact implies that $C(l_1, (l_\infty)^*)$ is not proximinal in $L(l_1, (l_\infty)^*)$. By Proposition 3 it is the same thing as Theorem 1 claims.

PROPOSITION 6. *For every set Γ, $C(l_1, l_1(\Gamma))$ is proximinal in $L(l_1, l_1(\Gamma))$.*

Proof. Let $T \in L(l_1, l_1(\Gamma))$. The range of T is contained in a subspace G of $l_1(\Gamma)$ isometric to l_1 (unless Γ is finite) such that there is a contractive projection of $l_1(\Gamma)$ onto G. But $C(l_1, l_1)$ is proximinal in $L(l_1, l_1)$ (e.g., [9]) and this yields the result easily.

Proof of Theorem 3. The "if" part is Proposition 6 since then $L_1(\mu) = l_1(\Gamma)$ for some Γ. For the "only if" part: by the proof of Theorem 1,

$C(l_1, L_1(0, 1))$ is not proximinal in $L(l_1, L_1(0, 1))$. By Proposition 5, $C(l_1, L_1(\mu))$ is not proximinal in $L(l_1, L_1(\mu))$ whenever $L_1(0, 1)$ is isometric to the range of a contractive projection in $L_1(\mu)$. But when μ is non-purely atomic, this is the case (see [8, Sect. 14]).

APPENDIX

Y. Benyamini has obtained an interesting consequence of the results of this paper. We reproduce it here. First we need

LEMMA. *Let X be a separable subspace of l_∞. Then there is a subspace E of l_∞, $X \subset E \subset l_\infty$ so that E is isometric to a range of a contractive projection in* $C(0, 1)$.

Proof. We assume the scalar field is complex; for the real case an obvious modification yields the same. Denote by E the (separable commutative) algebra with involution generated in l_∞ by X and the unit $(1, 1,...)$. By the Gelfand–Naimark theorem (see [12]), E is isometric to some $C(S)$ with S a compact metric space (E is separable—thus S is metric). By Milutin's lemma (see [12]), $C(S)$ is isometric to the range of a contractive projection in $C(0, 1)$.

THEOREM (Benyamini). $C(C(0, 1), C(0, 1))$ *is not proximinal in* $L(C(0, 1), C(0, 1))$.

Proof. $L_1(0, 1)$ is isometric to the range of a contractive projection in $C(0, 1)^*$ (see [8]). Hence, as in the proof of Theorem 1, $C(l_1, C(0, 1)^*)$ is not proximinal in $L(l_1, C(0, 1)^*)$. By Proposition 3, $C(C(0, 1), l_\infty)$ is not proximinal in $L(C(0, 1), l_\infty)$.

Let T: $C(0, 1) \to l_\infty$ be an operator which has no closest compact operator. There is a sequence $K_n : C(0, 1) \to l_\infty$ of compact operators such that $\| T - K_n \| \to \| T \|_e$. Denote by X the (separable!) subspace of l_∞ generated by the images of T and all the K_n's. Define E as in the lemma. Since

$$d(T, C(C(0, 1), l_\infty)) = d(T, C(C(0, 1), E))$$

there is no closest compact operator to T in $C(C(0, 1), E)$. By Proposition 5, $C(C(0, 1), C(0, 1))$ is not proximinal in $L(C(0, 1), C(0, 1))$.

Remark. This is the first known example of a separable classical Banach space E for which $C(E, E)$ is not proximinal in $L(E, E)$.

A. Lima has kindly pointed out to us that since l_1 is isometric to the range of a contractive projection in $L_1(0, 1)$, then Theorem 3 implies that $C(L_1(0, 1), L_1(0, 1))$ is not proximinal in $L(L_1(0, 1), L_1(0, 1))$.

Finally, let us note that since l_∞ is isometric to the range of a contractive projection on $L_\infty(0, 1)$, it follows from Theorem 1 that there is an operator $T: L_\infty(0, 1) \to L_\infty(0, 1)$ with no best compact approximation $K: L_\infty(0, 1) \to L_\infty(0, 1)$.

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